

RADIAL LENGTH, RADIAL JOHN DISKS AND K-QUASICONFORMAL HARMONIC MAPPINGS

SHAOLIN CHEN AND SAMINATHAN PONNUSAMY

ABSTRACT. In this article, we continue our investigations of the boundary behavior of harmonic mappings. We first discuss the classical problem on the growth of radial length and obtain a sharp growth theorem of the radial length of K -quasiconformal harmonic mappings. Then we present an alternate characterization of radial John disks. In addition, we investigate the linear measure distortion and the Lipschitz continuity on K -quasiconformal harmonic mappings of the unit disk onto a radial John disk. Finally, using Pommerenke interior domains, we characterize certain differential properties of K -quasiconformal harmonic mappings

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

This paper continues the study of previous work of the authors [6] and is mainly motivated by the articles of Beardon and Carne [3], Carroll and Twomey [4], Chuaqui et al. [10], Pommerenke [29], and the monograph of Pommerenke [30]. In order to state our first result concerning the growth of the *radial length* of K -quasiconformal harmonic mappings (see Theorem 1), we need to recall some basic definitions and some results which motivate the present work.

Let f be a complex-valued and continuously differentiable function defined in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and let $\ell_f(\theta, r)$ be the length of the f -image (with counting multiplicity) of the *radial line segment* $[0, z]$ from 0 to $z = re^{i\theta} \in \mathbb{D}$, where $\theta \in [0, 2\pi]$ is fixed and $r \in [0, 1)$. Then (cf. [5])

$$\ell_f(\theta, r) := \ell(f([0, z])) = \int_0^r |df(\rho e^{i\theta})| = \int_0^r |f_z(\rho e^{i\theta}) + e^{-2i\theta} f_{\bar{z}}(\rho e^{i\theta})| d\rho.$$

In [21], Keogh showed that if f is a bounded, analytic and univalent function in \mathbb{D} , then, for each $\theta \in [0, 2\pi]$,

$$(1.1) \quad \ell_f(\theta, r) = O\left(\left(\log(1/(1-r))\right)^{1/2}\right) \quad \text{as } r \rightarrow 1^-.$$

Throughout the discussion, we let

$$(1.2) \quad \psi(r) = \left(\log(1/(1-r))\right)^{1/2} \quad \text{for } 0 < r < 1.$$

File: Ch-P-Kqc-2016_final.tex, printed: 27-1-2017, 1.42

2010 *Mathematics Subject Classification.* Primary: 30C62, 30C75; Secondary: 30C20, 30C25, 30C45, 30F45, 30H10.

Key words and phrases. K -quasiconformal harmonic mapping, radial John disk, radial length, Pommerenke interior domain.

This second author is on leave from IIT Madras.

Keogh also gave some examples to show that the exponent $1/2$ in (1.1) can not be decreased. Jenkins improved on these examples in [16], and Kennedy [20] presented further examples by showing that

$$\ell_f(\theta, r) = O(\mu(r)\psi(r)) \text{ as } r \rightarrow 1^-$$

is false in general for every positive function μ in $[0, 1)$ satisfying $\mu(r) \rightarrow 0$ as $r \rightarrow 1^-$. In [4], Carroll and Twomey established certain refinements and extension of these results without the boundedness condition in the following form.

Theorem A. *Suppose that $f(z) = a_1z + a_2z^2 + \cdots$ is univalent in \mathbb{D} . Then, for any fixed $\theta \in [0, 2\pi]$, there is a constant $C_1 > 0$ such that*

$$(1.3) \quad \ell_f(\theta, r) \leq C_1 \max_{\rho \in [0, r]} |f(\rho e^{i\theta})| \psi(r) \text{ for } r \in (0.5, 1).$$

If, further, $f(re^{i\theta}) = O(1)$ as $r \rightarrow 1^-$, then (1.1) holds.

Later, Beardon and Carne [3] gave a relatively simple argument to Theorem A in hyperbolic geometry and provided with further examples. It is worth pointing out here two results which strengthened (1.3) and was inspired by the work of Sheil-Small [33] and Hall [14]. If $f \in \mathcal{S}$ is starlike, i.e. $f(\mathbb{D})$ contains the line segment $[0, w]$ whenever it contains w , then (see [19])

$$\ell_f(\theta, r) \leq |f(re^{i\theta})|(1+r) < 2|f(re^{i\theta})| \text{ for } r \in (0, 1)$$

and the inequality of course is not sharp for all r , but the bound 2 sharp as the Koebe function $k(z) = z/(1-z)^2$ shows and is attained when r approaches 1 (see [14, 33]). Later in 1993, Balasubramanian et al. [1] showed that if $f \in \mathcal{S}$ is convex, i.e. $f(\mathbb{D})$ is a convex domain, then

$$\ell_f(\theta, r) \leq |f(re^{i\theta})| r^{-1} \arcsin r \text{ for } r \in (0, 1)$$

and the inequality is sharp as the convex function $f(z) = z/(1-z)$ shows. Note that $\varphi(r) = r^{-1} \arcsin r$ is increasing on $(0, 1)$ and $\varphi(r) \leq \lim_{r \rightarrow 1^-} \varphi(r) = \pi/2$ and thus, the conjecture of Hall [15] was settled (see also [2]).

The first aims of this paper is to extend Theorem A for the case of harmonic quasiconformal mappings (see Theorem 1 below). We need some preparation to state this result.

For a real 2×2 matrix A , we use the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$. For $z = x + iy \in \mathbb{C}$, the formal derivative of the complex-valued function $f = u + iv$ is given by the Jacobian matrix

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that

$$\|D_f\| = |f_z| + |f_{\bar{z}}| \text{ and } l(D_f) = ||f_z| - |f_{\bar{z}}||,$$

where $f_z = (1/2)(f_x - if_y)$ and $f_{\bar{z}} = (1/2)(f_x + if_y)$. Let Ω be a domain in \mathbb{C} , with non-empty boundary. A sense-preserving homeomorphism f from a domain Ω

onto Ω' , contained in the Sobolev class $W_{loc}^{1,2}(\Omega)$, is said to be a K -quasiconformal mapping if, for $z \in \Omega$,

$$\|D_f(z)\|^2 \leq K |\det D_f(z)|, \text{ i.e., } \|D_f(z)\| \leq Kl(D_f(z)),$$

where $K \geq 1$ and $\det D_f$ is the determinant of D_f (cf. [18, 22, 35, 36]).

Let \mathcal{S}_H denote the family of sense-preserving planar harmonic univalent mappings $f = h + \bar{g}$ in \mathbb{D} , with the normalization $h(0) = g(0) = 0$ and $h'(0) = 1$. Recall that f is sense-preserving if the Jacobian J_f of f given by

$$J_f := \det D_f = |f_z|^2 - |f_{\bar{z}}|^2 = |h'|^2 - |g'|^2$$

is positive. Thus, f is locally univalent and sense-preserving in \mathbb{D} if and only if $J_f(z) > 0$ in \mathbb{D} ; or equivalently if $h' \neq 0$ in \mathbb{D} and the dilatation $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ in \mathbb{D} (see [11, 12, 23]). The family \mathcal{S}_H together with a few other geometric subclasses, originally investigated in detail by [11, 34], became instrumental in the study of univalent harmonic mappings (see [12, 31]) and has attracted the attention of many function theorists. If the co-analytic part g is identically zero in the decomposition of $f = h + \bar{g}$, then the class \mathcal{S}_H reduces to the classical family \mathcal{S} of all normalized analytic univalent functions $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in \mathbb{D} . If $\mathcal{S}_H^0 = \{f = h + \bar{g} \in \mathcal{S}_H : g'(0) = 0\}$, then the family \mathcal{S}_H^0 is both normal and compact. See [11] and also [8, 6, 12, 31].

Theorem 1. *For $K \geq 1$, let $f \in \mathcal{S}_H$ be a K -quasiconformal harmonic mapping. Then, for any fixed $\theta \in [0, 2\pi]$, there is a constant $C_2 > 0$ such that*

$$\ell_f(\theta, r) \leq C_2 \max_{\rho \in [0, r]} |f(\rho e^{i\theta})| \psi(r) \text{ for } r \in (0.5, 1).$$

If, further, $f(re^{i\theta}) = O(1)$ as $r \rightarrow 1^-$, then

$$\ell_f(\theta, r) = O(\psi(r)) \text{ as } r \rightarrow 1^-,$$

and the exponent $1/2$ in $\psi(r)$ defined by (1.2) cannot be replaced by a smaller number.

First we remark that if $K = 1$, then Theorem 1 coincides with Theorem A. Secondly, the proof of Theorem 1 is substantially harder than the proof of Theorem A. This is because Beardon and Carne's argument of Theorem A in [3] is not applicable in the proof of Theorem 1.

We need further notation and terminology before stating our second result. Let $d_{\Omega}(z)$ be the Euclidean distance from z to the boundary $\partial\Omega$ of Ω . If $\Omega = \mathbb{D}$, then we set $d(z) := d_{\mathbb{D}}(z)$.

Definition 1. A bounded simply connected plane domain G is called a c -John disk for $c \geq 1$ with John center $w_0 \in G$ if for each $w_1 \in G$ there is a rectifiable arc γ , called a John curve, in G with end points w_1 and w_0 such that

$$\sigma_{\ell}(w) \leq cd_G(w)$$

for all w on γ , where $\gamma[w_1, w]$ is the subarc of γ between w_1 and w , and $\sigma_{\ell}(w)$ is the Euclidean length of $\gamma[w_1, w]$ (see [6, 13, 17, 28, 30]).

Remark 1. If f is a complex-valued and univalent mapping in \mathbb{D} , $G = f(\mathbb{D})$ and, for $z \in \mathbb{D}$, $\gamma = f([0, z])$ in Definition 1, then we call c -John disk a *radial c -John disk*, where $w_0 = f(0)$ and $w = f(z)$. In particular, if f is a conformal mapping, then we call c -John disk a *hyperbolic c -John disk*. It is well known that any point $w_0 \in G$ can be chosen as a John center by modifying the constant c if necessary. When we do not wish to emphasize the role of c , then we regard the c -John disk simply as a John disk in the natural way (cf. [6, 13, 17, 28]).

Unless otherwise stated, throughout the discussion we consider the following terminology. Denote by $\mathcal{F}(K)$ if $f \in \mathcal{F}$ and is a K -quasiconformal harmonic mapping in \mathbb{D} , where $K \geq 1$. Also, we denote by $\mathcal{F}(K, \Omega)$ if $f \in \mathcal{F}(K)$ and f maps \mathbb{D} onto Ω . We prove several results mainly when \mathcal{F} equals one of \mathcal{S}_H , \mathcal{S}_H^0 , and \mathcal{S}_{H_2} , and Ω equals either radial John disk or Pommerenke interior domain.

Further, for $z \in \mathbb{D}$, we define

$$(1.4) \quad B(z) := \{\zeta : |z| \leq |\zeta| < 1, |\arg z - \arg \zeta| \leq \pi(1 - |z|)\}.$$

In the following, we continue our previous work of [6], and give another characterization of the radial John disk.

Theorem 2. *Let $f \in \mathcal{S}_H^0(K)$. Then the following are equivalent:*

- (i) $\Omega = f(\mathbb{D})$ is a radial John disk.
- (ii) There is an $x \in (0, 1)$ such that

$$\sup_{|\zeta|=1} \sup_{r \in (0,1)} \frac{(1 - \rho^2) \|D_f(\rho\zeta)\|}{(1 - r^2) \|D_f(r\zeta)\|} < 1 \text{ for } \rho = \frac{x + r}{1 + xr}.$$

$$(iii) \quad \sup_{z \in \mathbb{D}, w \in B(z)} \frac{|f(z) - f(w)|}{(1 - |z|^2) \|D_f(z)\|} < \infty.$$

Next, we establish the linear measure distortion on K -quasiconformal harmonic mappings of \mathbb{D} onto a radial John disk.

Theorem 3. *Let $f = h + \bar{g} \in \mathcal{S}_H^0(K, \Omega)$, where Ω is a radial John disk. Then, for $a_1, a_2 \in \mathbb{D}$ with $B(a_1) \subset B(a_2)$, there is a positive constant C_3 such that*

$$\frac{\text{diam} f(B(a_1))}{\text{diam} f(B(a_2))} \leq C_3 \left(\frac{\ell(B(a_1) \cap \partial\mathbb{D})}{\ell(B(a_2) \cap \partial\mathbb{D})} \right)^\alpha.$$

where $\alpha = \sup_{f \in \mathcal{S}_H} \frac{|h''(0)|}{2}$ and $B(z)$ is defined by (1.4).

We remark that $2 \leq \alpha = \sup_{f \in \mathcal{S}_H} \frac{|h''(0)|}{2} < \infty$, but the sharp value of α is still unknown (see [6, 9, 12, 34]). We discuss the Lipschitz continuity on K -quasiconformal harmonic mappings of \mathbb{D} onto a radial John disk, which is as follows.

Theorem 4. *Let $f = h + \bar{g} \in \mathcal{S}_H^0(K, \Omega)$, where Ω is a radial John disk. Then, for $z \in \mathbb{D}$ with $|z| \geq \frac{1}{2}$ and $\zeta_1, \zeta_2 \in B(z)$, there are constants $\delta_1 \in (0, 1)$ and $C_4 > 0$ such that*

$$|f(\zeta_1) - f(\zeta_2)| \leq C_4 d_\Omega(f(z)) \left(\frac{|\zeta_1 - \zeta_2|}{1 - |z|} \right)^{\delta_1}.$$

Let $f \in \mathcal{S}_H(K, G)$, where G is domain. For $0 < r < 1$, let $\mathbb{D}_r = \{z : |z| < r\}$ and $\partial\mathbb{D}_r$ denote the boundary of \mathbb{D}_r . Now, for $w_1, w_2 \in f(\partial\mathbb{D}_r)$, let γ_r be the smaller subarc of $f(\partial\mathbb{D}_r)$ between w_1 and w_2 , and let

$$d_{G_r}(w_1, w_2) = \inf_{\Gamma} \text{diam} \Gamma,$$

where Γ runs through all arcs from w_1 to w_2 that lie in $G_r = f(\mathbb{D}_r)$ except for their endpoints. If

$$(1.5) \quad \sup_{0 < r < 1} \left\{ \sup_{w_1, w_2 \in \gamma_r} \frac{\ell(\gamma_r[w_1, w_2])}{d_{G_r}(w_1, w_2)} \right\} < \infty,$$

then we call G a *Pommerenke interior domain* (cf. [6, 29]). In particular, if G is bounded, then we call G as a *bounded Pommerenke interior domain*.

Given a sense-preserving harmonic mapping $f = h + \bar{g}$ in \mathbb{D} , fix $\zeta \in \mathbb{D}$ and perform a disk automorphism (also called Koebe transform F of f) to obtain

$$(1.6) \quad F(z) = \frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{h'(\zeta)(1-|\zeta|^2)} =: H(z) + \overline{G(z)}.$$

A calculation gives,

$$\frac{H''(0)}{2} = \frac{1}{2} \left\{ (1-|\zeta|^2) \frac{h''(\zeta)}{h'(\zeta)} - 2\bar{\zeta} \right\}.$$

Now, we consider the class \mathcal{S}_{H_2} of all harmonic mappings $f = h + \bar{g} \in \mathcal{S}_H$ satisfying

$$(1.7) \quad \sup_{z \in \mathbb{D}} \left| (1-|z|^2) \frac{h''(z)}{h'(z)} - 2\bar{z} \right| < 4.$$

This inequality obviously holds if $h \in \mathcal{S}$ and h is not the Koebe function $z/(1-e^{i\theta}z)^2$, $\theta \in \mathbb{R}$. Note that for the Koebe function the supremum turns out to be 4. Our next two results are extension of [29, Theorem 3].

Theorem 5. *Let $f \in \mathcal{S}_{H_2}(K, G)$, where G is a bounded Pommerenke interior domain. If there are positive constants $\delta_2 \in (0, 1)$ and C_5 such that, for each $\zeta \in \partial\mathbb{D}$ and for $0 \leq \rho_1 \leq \rho_2 < 1$,*

$$(1.8) \quad \|D_f(\rho_2\zeta)\| \leq C_5 \left(\frac{1-\rho_2}{1-\rho_1} \right)^{\delta_2-1} \|D_f(\rho_1\zeta)\|,$$

then

$$\sup_{\zeta \in \mathbb{D}} \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{\|D_f(\xi)\|}{\|D_f(\zeta)\|} \frac{1-|\zeta|^2}{|\xi-\zeta|^2} |d\xi| < \infty.$$

We remark that if $K = 1$, then Theorem 5 coincides with [29, Theorem 3].

By using similar reasoning as in the proof of Theorem 5, one can easily get the following result which replaces the assumption $f \in \mathcal{S}_{H_2}$ by a more general condition $f \in \mathcal{S}_H$ and thus, we omit its proof.

Theorem 6. *Let $f \in \mathcal{S}_H(K, G)$, where G is a bounded Pommerenke interior domain. If there are constants $C_6 > 0$, $C_7 > 0$, $\delta_3 > 0$ and $\delta_4 \in (0, 1)$ such that, for each $\zeta \in \partial\mathbb{D}$ and for $0 \leq \rho_1 \leq \rho_2 < 1$,*

$$C_6 \left(\frac{1 - \rho_1}{1 - \rho_2} \right)^{\delta_3 - 1} \|D_f(\rho_1 \zeta)\| \leq \|D_f(\rho_2 \zeta)\| \leq C_7 \left(\frac{1 - \rho_2}{1 - \rho_1} \right)^{\delta_4 - 1} \|D_f(\rho_1 \zeta)\|,$$

then

$$\sup_{\zeta \in \mathbb{D}} \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{\|D_f(\xi)\|}{\|D_f(\zeta)\|} \frac{1 - |\zeta|^2}{|\xi - \zeta|^2} |d\xi| < \infty.$$

Also, the following result easily follows from Theorem 5 and [6, Theorem 1].

Corollary 1. *For $K \geq 1$, let $f \in \mathcal{S}_{H_2} \cap \mathcal{S}_H^0$ be a K -quasiconformal harmonic mapping from \mathbb{D} onto a bounded Pommerenke interior domain G . If G is a radial John disk, then*

$$\sup_{\zeta \in \mathbb{D}} \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{\|D_f(\xi)\|}{\|D_f(\zeta)\|} \frac{1 - |\zeta|^2}{|\xi - \zeta|^2} |d\xi| < \infty.$$

The proofs of Theorems 1-5 will be presented in Section 2.

2. THE PROOFS OF THE MAIN RESULTS

Let $\lambda_{\mathbb{D}}$ stand for the *hyperbolic distance* (or *Poincaré distance*) on the unit disk \mathbb{D} . We have

$$\lambda_{\mathbb{D}}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1 - |z|^2} = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|,$$

where the infimum is taken over all smooth curves γ in \mathbb{D} connecting $z_1 \in \mathbb{D}$ and $z_2 \in \mathbb{D}$ (cf. [30]). In [34], Sheil-Small proved that if $f = h + \bar{g} \in \mathcal{S}_H$, then

$$(2.1) \quad \frac{(1 - |z|)^{\alpha-1}}{(1 + |z|)^{\alpha+1}} \leq |h'(z)| \leq \frac{(1 + |z|)^{\alpha-1}}{(1 - |z|)^{\alpha+1}}$$

and

$$\alpha := \sup_{f \in \mathcal{S}_H} \frac{|h''(0)|}{2} < \infty.$$

Unless otherwise stated, the number α will be used throughout the discussion and is indeed called the order of the linear invariant family \mathcal{S}_H (see [34]).

Lemma 1. *Suppose that $f \in \mathcal{S}_H(K)$. Then, for $z_0, z_1 \in \mathbb{D}$,*

$$\frac{1}{\alpha(1 + K)} [1 - e^{-2\alpha\lambda_{\mathbb{D}}(z_1, z_0)}] \leq \frac{|f(z_1) - f(z_0)|}{(1 - |z_0|^2)|f_z(z_0)|} \leq \frac{K}{\alpha(1 + K)} [e^{2\alpha\lambda_{\mathbb{D}}(z_1, z_0)} - 1].$$

In particular,

$$(2.2) \quad \frac{1}{\alpha(1 + K)} [1 - e^{-2\alpha\lambda_{\mathbb{D}}(z, 0)}] \leq |f(z)| \leq \frac{K}{\alpha(1 + K)} [e^{2\alpha\lambda_{\mathbb{D}}(z, 0)} - 1], \quad z \in \mathbb{D}.$$

Proof. By assumption $f = h + \bar{g} \in \mathcal{S}_H$ is a K -quasiconformal harmonic mapping, where h and g are analytic in \mathbb{D} . Thus, by (2.1), we have

$$(2.3) \quad \|D_f(z)\| \leq \frac{2K}{K+1} |h'(z)| \leq \frac{2K}{K+1} \frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}$$

and thus, for $z \in \mathbb{D}$, we obtain

$$(2.4) \quad \begin{aligned} |f(z)| &\leq \int_{[0,z]} \|D_f(\zeta)\| |d\zeta| \\ &\leq \frac{2K}{K+1} \int_0^{|z|} \frac{(1+\rho)^{\alpha-1}}{(1-\rho)^{\alpha+1}} d\rho = \frac{K}{\alpha(K+1)} \left[\left(\frac{1+|z|}{1-|z|} \right)^\alpha - 1 \right]. \end{aligned}$$

On the other hand, let Γ be the preimage under f of the radial segment from 0 to $f(z)$. Again, because

$$l(D_f(z)) \geq \frac{2}{K+1} |h'(z)| \geq \frac{2}{K+1} \frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}},$$

it follows that

$$(2.5) \quad |f(z)| \geq \int_\Gamma l(D_f(\zeta)) |d\zeta| \geq \frac{1}{\alpha(K+1)} \left[1 - \left(\frac{1-|z|}{1+|z|} \right)^\alpha \right].$$

Let $z = \frac{z_1 - z_0}{1 - \bar{z}_0 z_1}$ so that $z_1 = \frac{z + z_0}{1 + \bar{z}_0 z}$, where $z_0, z_1 \in \mathbb{D}$. Then, by assumption,

$$F(z) = \frac{f(z) - f(z_0)}{(1 - |z_0|^2)h'(z_0)} \in \mathcal{S}_H$$

and is a K -quasiconformal harmonic mapping, i.e. $F \in \mathcal{S}_H(K)$. Applying (2.4) and (2.5) to F gives us the desired result if we take into account of the fact that

$$\left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| = \frac{e^{2\lambda_{\mathbb{D}}(z_1, z_2)} - 1}{e^{2\lambda_{\mathbb{D}}(z_1, z_2)} + 1} = \tanh \lambda_{\mathbb{D}}(z_1, z_2).$$

The proof of the lemma is complete. \square

Lemma 2. Assume that $f \in \mathcal{S}_H(K)$. Then

$$\|D_f(z)\| |z| \leq \frac{C_8 |f(z)|}{1 - |z|} \quad \text{for } z \in \mathbb{D},$$

where

$$(2.6) \quad C_8 = 2\alpha K \sup_{z \in \mathbb{D}} \left\{ \frac{|z|(1+|z|)^{\alpha-1}}{[(1+|z|)^\alpha - (1-|z|)^\alpha]} \right\} \geq K.$$

Proof. Suppose that $f = h + \bar{g} \in \mathcal{S}_H(K)$, where h and g are analytic in \mathbb{D} . Next, for fixed $\zeta \in \mathbb{D}$, consider the Koebe transform F of f given by (1.6). By assumption, $F \in \mathcal{S}_H$ and is also a K -quasiconformal harmonic mapping. By letting $z = -\zeta$ in (1.6) and applying (2.2) to F , we obtain (since $f(0) = 0$)

$$|F(-\zeta)| = \frac{|f(\zeta)|}{(1 - |\zeta|^2)|h'(\zeta)|} \geq \frac{1}{\alpha(1+K)} \frac{[(1+|\zeta|)^\alpha - (1-|\zeta|)^\alpha]}{(1+|\zeta|)^\alpha}$$

which gives

$$\frac{|h'(\zeta)|}{(1+K)|f(\zeta)|} \leq \frac{(1+|\zeta|)^{\alpha-1}}{[(1+|\zeta|)^\alpha - (1-|\zeta|)^\alpha]} \cdot \frac{\alpha}{1-|\zeta|}.$$

Since this follows for each $\zeta \in \mathbb{D}$, by the first inequality in (2.3), we easily have

$$\frac{\|D_f(z)\| |z|}{|f(z)|} \leq \frac{2K}{K+1} \frac{|h'(z)| |z|}{|f(z)|} \leq \frac{C_8}{1-|z|},$$

where C_8 is given by (2.6). □

Lemma 3. *Let $f \in \mathcal{S}_H(K)$ and, for any fixed $\theta \in [0, 2\pi]$, set*

$$m_f(r, \theta) = \max_{\rho \in [0, r]} |f(\rho e^{i\theta})|,$$

where $r \in [0, 1)$. Then, for $0 < \rho_0 \leq r < 1$ and $0 \leq \rho \leq r$, there is a constant $C_9 > 0$ which depends only on ρ_0 such that

$$(2.7) \quad \frac{|f(\rho e^{i\theta})|}{\rho} \leq C_9 m_f(r, \theta),$$

where ρ_0 is a constant.

Proof. Without loss of generality, we assume that $\theta = 0$. Clearly, (2.2) yields that

$$\lim_{\rho \rightarrow 0^+} \frac{|f(\rho)|}{\rho} \leq \frac{K}{\alpha(1+K)} \lim_{\rho \rightarrow 0^+} \frac{e^{2\alpha\lambda_{\mathbb{D}}(\rho, 0)} - 1}{\rho} = \frac{2K}{1+K},$$

which implies that $f(\rho)/\rho$ is bounded in $[0, \rho_0]$, where ρ_0 is a constant such that $0 < \rho_0 \leq r < 1$. Hence there is a constant $C_{10} > 0$ such that

$$(2.8) \quad \frac{f(\rho)}{\rho} \leq C_{10} m_f(r, 0) \text{ for } \rho \in [0, \rho_0],$$

where $r \in [\rho_0, 1)$. For $r \in [0, 1)$, let

$$T(r) = \frac{K}{\alpha(1+K)} [1 - e^{-2\alpha\lambda_{\mathbb{D}}(r, 0)}].$$

Then T is increasing in $[0, 1)$, which, together with (2.2), yields that

$$(2.9) \quad 0 < T(\rho_0) \leq T(\rho) \leq f(\rho) \leq \frac{f(\rho)}{\rho} \leq \frac{f(\rho)}{\rho_0} \leq \frac{1}{\rho_0} m_f(r, 0) \text{ for } \rho \in [\rho_0, r],$$

where $r \in [\rho_0, 1)$. Therefore, (2.7) follows from (2.8) and (2.9). □

Lemma 4. *For $r \in (0, 1)$, let Ω_r be the Stolz-type domain consisting of the interior of the convex hull of the point r and the disk $\mathbb{D}_{r/4}$. Then, for $z = \rho e^{i\eta} \in \Omega_r \setminus \mathbb{D}_{r/4}$,*

$$|\eta| \leq \frac{4\pi}{r\sqrt{15}}(r - \rho) < \frac{4\pi}{r\sqrt{15}}(1 - \rho).$$

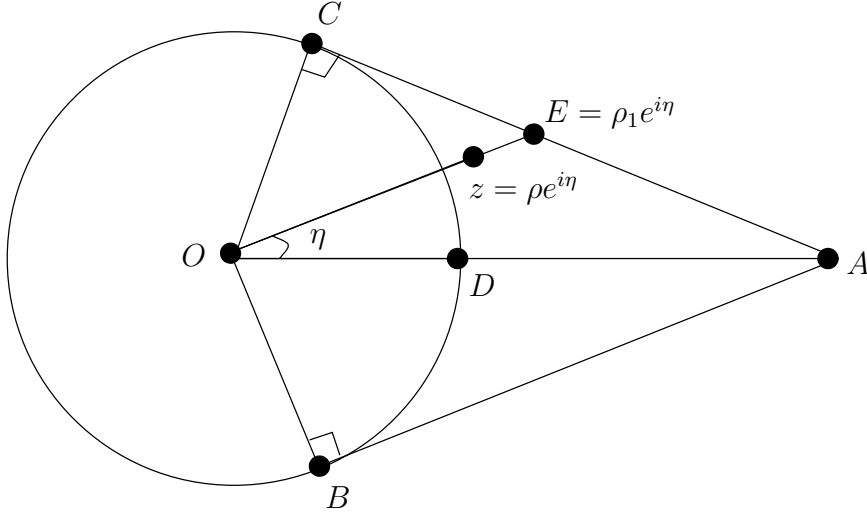


FIGURE 1. Stolz-type domain

Proof. Assume without loss of generality that $\eta \geq 0$. Let A , D , and E represent the points r , $r/4$, and $\rho_1 e^{i\eta}$ (see Figure 1), respectively. As $\angle OCA = \pi/2$, it is clear that

$$\sin \angle COA = \frac{\sqrt{15}}{4}, \quad \cos \angle COA = \frac{1}{4}, \quad \sin \angle COE = \frac{\sqrt{\rho_1^2 - \frac{r^2}{16}}}{\rho_1} \quad \text{and} \quad \cos \angle COE = \frac{r}{4\rho_1}.$$

Then, because $|(\sin \eta)/\eta| \geq 2/\pi$ for $|\eta| < \pi/2$, it follows that for $\eta \geq 0$

$$\begin{aligned} \frac{2\eta}{\pi} \leq \sin \eta &= \sin(\angle COA - \angle COE) \\ &= \sin \angle COA \cos \angle COE - \cos \angle COA \sin \angle COE \\ &= \frac{\sqrt{15}r^2 - \sqrt{16\rho_1^2 - r^2}}{16\rho_1} \\ &= \frac{r^2 - \rho_1^2}{\rho_1 (\sqrt{15}r + \sqrt{16\rho_1^2 - r^2})} \end{aligned}$$

Note that $\frac{r}{4} < \rho < \rho_1 < r$ and, because

$$\frac{r^2 - \rho_1^2}{\rho_1} < \frac{4}{r} (r^2 - \rho_1^2) < 8(r - \rho_1) < 8(r - \rho),$$

the last above relation clearly implies that

$$\frac{2\eta}{\pi} < \frac{8(r - \rho)}{r\sqrt{15}}$$

which gives the desired conclusion. Observe that $\frac{8}{r\sqrt{15}}(r - \rho)$ is less than $6/\sqrt{15}$ from which we also deduce that $|\eta| < 3\pi/\sqrt{15}$. \square

Proof of Theorem 1. Assume without loss of generality that $\theta = 0$. For $r \in (0, 1)$, we use Ω_r to denote the Stolz-type domain, where Ω_r is same as in Lemma 4. Let $z = \rho e^{i\eta} \in \Omega_r \setminus \mathbb{D}_{r/4}$. Then, by Lemma 4, there is a constant $C_{11} > 0$ which depends only on r such that

$$(2.10) \quad |\eta| < C_{11}(1 - \rho).$$

Suppose that $f = h + \bar{g} \in \mathcal{S}_H(K)$. By calculations, we get

$$(2.11) \quad \log \frac{f(\rho e^{i\eta})}{\rho e^{i\eta}} - \log \frac{f(\rho)}{\rho} = i \int_0^\eta \left(\frac{\rho e^{it} h'(\rho e^{it}) - \rho e^{-it} \overline{g'(\rho e^{it})}}{f(\rho e^{it})} - 1 \right) dt.$$

Taking real part of (2.11) on both sides, and then using (2.10), (2.11) and Lemma 2, we see that there is a constant C_{12} such that

$$\begin{aligned} \log \frac{|f(\rho e^{i\eta})|}{\rho} - \log \frac{|f(\rho)|}{\rho} &\leq \int_0^\eta \frac{\rho \|D_f(\rho e^{it})\|}{|f(\rho e^{it})|} dt \\ &\leq C_{12} \int_0^\eta \frac{dt}{1 - \rho} \leq C_{11} C_{12}, \end{aligned}$$

which gives that

$$(2.12) \quad |f(z)| = |f(\rho e^{i\eta})| \leq e^{C_{11} C_{12}} |f(\rho)|.$$

By (2.2), we see that (2.12) also holds for $z \in \mathbb{D}_{r/4}$. Then, by (2.12), there is constant C_{13} such that $f(\Omega_r)$ is contained in $\mathbb{D}_{C_{13}m_f(r,0)}$, which yields that

$$(2.13) \quad \int_{\Omega_r} J_f(\zeta) dA(\zeta) \leq C_{13}^2 m_f^2(r, 0),$$

where $\zeta = x + iy$, $dA = dxdy/\pi$ and $m_f(r, \theta)$ is defined as in Lemma 3.

By [24, Theorem 2], there is a constant C_{14} such that

$$(2.14) \quad \int_0^1 (1 - \rho) |H'(\rho)|^2 d\rho \leq C_{14} \int_{\Omega_1} |H'(z)|^2 dA(z),$$

where $H(z)$ is analytic in \mathbb{D} . For $r \in (0, 1)$, let $H(z) = h(rz)$ for $z \in \mathbb{D}$. Then, by (2.14), we obtain

$$\int_0^r (r - \rho) |h'(\rho)|^2 d\rho \leq C_{15} \int_{\Omega_r} |h'(z)|^2 dA(z),$$

which implies that

$$\begin{aligned} \int_0^r (r - \rho) \|D_f(\rho)\|^2 d\rho &\leq C_{15} K \int_{\Omega_r} J_f(z) dA(z), \\ (2.15) \quad &\leq C_{13}^2 C_{15} K m_f^2(r, 0) \quad (\text{by (2.13)}), \end{aligned}$$

where $C_{15} > 0$ is a constant.

By Lemmas 2 and 3, for $r \in (1/2, 1)$ and $\rho \in [0, r]$, there is a constant C_{16} such that

$$(2.16) \quad \int_0^r \|D_f(\rho)\|^2 d\rho \leq C_{16} \int_0^r \frac{m_f^2(r, 0)}{(1 - \rho)^2} d\rho = C_{16} m_f^2(r, 0) \frac{r}{1 - r}.$$

Writing $1 - \rho = (1 - r) + (r - \rho)$ and then, applying (2.15) and (2.16), it follows that

$$(2.17) \quad \int_0^r (1 - \rho) \|D_f(\rho)\|^2 d\rho \leq (C_{13}^2 C_{15} K + C_{16}) m_f^2(r, 0).$$

Therefore, by (2.17), we conclude that

$$\begin{aligned} \ell_f(0, r) &\leq \int_0^r \|D_f(\rho)\| d\rho \\ &\leq \left(\int_0^r (1 - \rho) \|D_f(\rho)\|^2 d\rho \right)^{\frac{1}{2}} \left(\int_0^r \frac{d\rho}{1 - \rho} \right)^{\frac{1}{2}} \\ &\leq (C_{13}^2 C_{15} K + C_{16})^{1/2} m_f(r, 0) \left(\log \frac{1}{1 - r} \right)^{\frac{1}{2}}. \end{aligned}$$

Now we prove the sharpness part. For any $\tau \in (0, 1/2)$, by [16, 21], there is a function $h_0 \in \mathcal{S}$ such that,

$$(2.18) \quad \ell_{h_0}(0, r) > C_{17} \left(\log \frac{1}{1 - r} \right)^\tau \text{ as } r \rightarrow 1^-,$$

where C_{17} is a positive constant. Finally, consider

$$f_0(z) = h_0(z) + \frac{K - 1}{K + 1} \overline{h_0(z)}, \quad z \in \mathbb{D},$$

and observe that $f_0 \in \mathcal{S}_H$ and is a K -quasiconformal harmonic mapping. Consequently,

$$\ell_{f_0}(0, r) = \int_0^r \left| h_0'(\rho) + \frac{K - 1}{K + 1} \overline{h_0'(\rho)} \right| d\rho \geq \frac{2}{K + 1} \int_0^r |h_0'(\rho)| d\rho = \frac{2}{K + 1} \ell_{h_0}(0, r),$$

which, together with (2.18), implies that

$$\ell_{f_0}(0, r) > \frac{2C_{17}}{K + 1} \left(\log \frac{1}{1 - r} \right)^\tau \text{ as } r \rightarrow 1^-.$$

The proof of this theorem is complete. \square

Lemma 5. *Let $f \in \mathcal{S}_H^0$. Then, for $\xi \in \partial\mathbb{D}$ and $0 \leq \rho \leq r < 1$,*

$$(2.19) \quad \frac{(1 - \rho^2) \|D_f(\rho\xi)\|}{(1 - r^2) \|D_f(r\xi)\|} \leq e^{2\alpha_{\lambda_{\mathbb{D}}(\rho, r)}}.$$

Proof. Let $f = h + \bar{g} \in \mathcal{S}_H^0$, where h and g are analytic in \mathbb{D} . For every $\mu \in \mathbb{D}$, consider the affine mapping

$$f_\mu = f + \mu \bar{f} = (h + \mu g) + \overline{(g + \mu h)}.$$

Clearly, $f_\mu \in \mathcal{S}_H$. For a fixed $\zeta \in \mathbb{D}$, we consider the Koebe transform F_μ of f_μ as given by (1.6). Then we can write $F_\mu = H_\mu + \overline{G_\mu}$ which again belongs to \mathcal{S}_H and obviously,

$$\frac{H_\mu''(0)}{2} = A_2(\zeta) = \frac{1}{2}(1 - |\zeta|^2) \frac{h''(\zeta) + \mu g''(\zeta)}{h'(\zeta) + \mu g'(\zeta)} - \bar{\zeta}.$$

Since $|A_2| \leq \alpha$, we see that

$$\begin{aligned} \left| \frac{\partial}{\partial \rho} \log [(1 - \rho^2)(h'(\rho\xi) + \mu g'(\rho\xi))] \right| &= \left| \frac{h''(\rho\xi) + \mu g''(\rho\xi)}{h'(\rho\xi) + \mu g'(\rho\xi)} - \frac{2\rho\bar{\xi}}{1 - \rho^2} \right| \\ &\leq \frac{2\alpha}{1 - \rho^2}, \end{aligned}$$

where $\xi \in \partial\mathbb{D}$. Integration leads to

$$\frac{(1 - r^2)|h'(r\xi) + \mu g'(r\xi)|}{(1 - \rho^2)|h'(\rho\xi) + \mu g'(\rho\xi)|} \geq \left(\frac{1 - r}{1 + r} \cdot \frac{1 + \rho}{1 - \rho} \right)^\alpha,$$

which gives

$$(1 - \rho^2)|h'(\rho\xi) + \mu g'(\rho\xi)| \leq e^{2\alpha\lambda_{\mathbb{D}}(\rho, r)}(1 - r^2)|h'(r\xi) + \mu g'(r\xi)|$$

and the desired inequality (2.19) follows from this and the arbitrariness of μ . \square

We remark that Mateljević [26] (see also [27, 25]) proved the following lemma for $f \in \mathcal{S}_H^0(K)$ instead of $f \in \mathcal{S}_H(K)$. That is, the normalization condition on f , namely, $f_{\bar{z}}(0) = 0$, is not necessary.

Lemma 6. *If $f \in \mathcal{S}_H(K)$ and $\Omega = f(\mathbb{D})$, then*

$$(2.20) \quad d_\Omega(f(z)) \geq \frac{\|D_f(z)\|(1 - |z|^2)}{16K} \quad \text{for } z \in \mathbb{D}.$$

Proof. Let $f = h + \bar{g} \in \mathcal{S}_H(K)$, where h and g are analytic in \mathbb{D} . Then the affine mapping f_0 defined by

$$f_0(z) = \frac{f(z) - \overline{g'(0)f(z)}}{1 - |g'(0)|^2}$$

belongs to \mathcal{S}_H^0 . By [11, Theorem 4.4], we have

$$(2.21) \quad \frac{|f(z)|}{1 - |g'(0)|} \geq |f_0(z)| = \frac{|f(z) - \overline{g'(0)f(z)}|}{1 - |g'(0)|^2} \geq \frac{|z|}{4(1 + |z|)^2}, \quad z \in \mathbb{D}.$$

Recall again, for any fixed $\zeta \in \mathbb{D}$, the Koebe transform F of f given by (1.6) belongs to \mathcal{S}_H and F is again a K -quasiconformal harmonic mapping. As a result, (2.21) applied to F shows that

$$\begin{aligned} \left| f\left(\frac{z + \zeta}{1 + \bar{\zeta}z}\right) - f(\zeta) \right| &\geq (1 - |\zeta|^2)|h'(\zeta)|(1 - |F_{\bar{z}}(0)|)\frac{|z|}{4(1 + |z|)^2} \\ &\geq (1 - |\zeta|^2)|h'(\zeta)|\left(\frac{2}{K + 1}\right)\frac{|z|}{4(1 + |z|)^2} \\ &\geq \frac{(1 - |\zeta|^2)\|D_f(\zeta)\|}{K}\frac{|z|}{4(1 + |z|)^2}, \end{aligned}$$

which implies that

$$d_\Omega(f(\zeta)) = \liminf_{|z| \rightarrow 1^-} \frac{\left| f\left(\frac{z + \zeta}{1 + \bar{\zeta}z}\right) - f(\zeta) \right|}{|z|} \geq \frac{\|D_f(\zeta)\|(1 - |\zeta|^2)}{16K}.$$

The proof of this Lemma is complete. \square

Lemma B. ([6, Lemma 2]) *Let a_1, a_2 and a_3 be positive constants and let $0 < |z_0| = 1 - \delta_5$, where $\delta_5 \in (0, 1)$. If $f \in \mathcal{S}_H$, $0 \leq 1 - a_2\delta_5 \leq |z| \leq 1 - a_1\delta_5$ and $|\arg z - \arg z_0| \leq a_3\delta_5$, then*

$$\frac{1}{M(a_1, a_2, a_3)} \|D_f(z_0)\| \leq \|D_f(z)\| \leq M(a_1, a_2, a_3) \|D_f(z_0)\|,$$

where $M(a_1, a_2, a_3) = 2e^{(1+\alpha)\left(a_3 + \frac{1}{2} \log \frac{2a_2 - a_1}{a_1}\right)}$.

Proof of Theorem 2. Let $f \in \mathcal{S}_H^0(K)$. First we show that (ii) \Rightarrow (i). We assume that

$$(2.22) \quad \frac{(1 - \rho^2) \|D_f(\rho\zeta)\|}{(1 - r^2) \|D_f(r\zeta)\|} \leq \beta < 1 \text{ for } \rho = \frac{x + r}{1 + xr}, |\zeta| = 1,$$

uniformly on r and ζ . Define $x_1 = x$ and x_k for $k = 2, 3, \dots$, by

$$\frac{1 + x_k}{1 - x_k} = \left(\frac{1 + x}{1 - x} \right)^k, \text{ i.e., } x_{k+1} = \frac{x + x_k}{1 + xx_k}.$$

Note that $\rho > r$ and thus, $x_{k+1} > x_k$. Consequently, by (2.22), we have

$$(2.23) \quad \frac{(1 - x_{k+1}^2) \|D_f(x_{k+1})\|}{(1 - x_k^2) \|D_f(x_k)\|} \leq \beta < 1.$$

Let $\delta_6 \in (0, 1)$ such that

$$\beta < \left(\frac{1 - x}{1 + x} \right)^{\delta_6}.$$

Then, for $j < k$, by (2.23),

$$\begin{aligned} \frac{(1 - x_k^2) \|D_f(x_k)\|}{(1 - x_j^2) \|D_f(x_j)\|} &= \frac{(1 - x_k^2) \|D_f(x_k)\|}{(1 - x_{k-1}^2) \|D_f(x_{k-1})\|} \\ &\times \frac{(1 - x_{k-1}^2) \|D_f(x_{k-1})\|}{(1 - x_{k-2}^2) \|D_f(x_{k-2})\|} \times \dots \times \frac{(1 - x_{j+1}^2) \|D_f(x_{j+1})\|}{(1 - x_j^2) \|D_f(x_j)\|} \\ &\leq \beta^{k-j} \\ &< \left(\frac{1 - x_k}{1 + x_k} \right)^{\delta_6} \left(\frac{1 - x_j}{1 + x_j} \right)^{-\delta_6} \\ (2.24) \quad &\leq \left(\frac{1 - x_k}{1 - x_j} \right)^{\delta_6}. \end{aligned}$$

By calculations, for $k = \{1, 2, \dots\}$,

$$\lambda_{\mathbb{D}}(x_k, x_{k+1}) = \lambda_{\mathbb{D}}(0, x),$$

which, together with (2.24) and Lemma 5, yields that there is a constant $C_{18} > 0$ such that

$$(2.25) \quad \frac{\|D_f(\rho\zeta)\|}{\|D_f(r\zeta)\|} \leq C_{18} \left(\frac{1-\rho}{1-r} \right)^{\delta_6-1}.$$

Hence, by (2.25) and [6, Theorem 1], we conclude that Ω is a radial John disk.

(i) \Rightarrow (ii). Suppose that $\Omega = f(\mathbb{D})$ is a radial John disk. Then, by [6, Theorem 1], there are constants $C_{19} > 0$ and $\delta_7 \in (0, 1)$ such that, for each $\zeta \in \partial\mathbb{D}$ and for $0 \leq \rho \leq r < 1$,

$$\frac{(1-\rho^2)\|D_f(\rho\zeta)\|}{(1-r^2)\|D_f(r\zeta)\|} \leq C_{19} \left(\frac{1-\rho}{1-r} \right)^{\delta_7} = C_{19} \left(\frac{1-x}{1+rx} \right)^{\delta_7} \leq C_{19}(1-x)^{\delta_7}.$$

It is not difficult to see that $C_{19}(1-x)^{\delta_7} < 1$ by taking x sufficiently close to 1.

Next we show that (i) \Rightarrow (iii). For $z = re^{i\theta} \in \mathbb{D}$ and $w = r_1e^{i\theta_1} \in B(z)$, by [6, Theorem 1] and Lemma B, we see that there are positive constants C_{20} , C_{21} and $\delta_8 \in (0, 1)$ such that

$$\begin{aligned} |f(z) - f(w)| &\leq |f(re^{i\theta}) - f(re^{i\theta_1})| + |f(re^{i\theta_1}) - f(r_1e^{i\theta_1})| \\ &\leq r \int_{\gamma'} \|D_f(re^{it})\| dt + \int_r^{r_1} \|D_f(\rho e^{i\theta_1})\| d\rho \\ &\leq C_{20}r \int_{\gamma'} \|D_f(re^{i\theta})\| dt + C_{21} \int_r^{r_1} \|D_f(re^{i\theta})\| \left(\frac{1-\rho}{1-r} \right)^{\delta_8-1} d\rho \\ &\leq C_{20}r \|D_f(re^{i\theta})\| \ell(\gamma') + \frac{C_{21}}{\delta_8} \|D_f(re^{i\theta})\| (1-r) \\ &\leq \left(2\pi C_{20} + \frac{C_{21}}{\delta_8} \right) \|D_f(re^{i\theta})\| (1-r), \end{aligned}$$

which gives that

$$\sup_{z \in \mathbb{D}, w \in B(z)} \frac{|f(z) - f(w)|}{(1-|z|^2)\|D_f(z)\|} < \infty,$$

where γ' is the smaller subarc of $\partial\mathbb{D}_r$ between $re^{i\theta}$ and $re^{i\theta_1}$.

Finally, we prove (iii) \Rightarrow (i). For $z \in \mathbb{D}$ and $w_1, w_2 \in B(z)$, there is a positive constant C_{22} such that

$$\begin{aligned} |f(w_1) - f(w_2)| &\leq |f(w_1) - f(z)| + |f(w_2) - f(z)| \\ &\leq C_{22}(1-|z|^2)\|D_f(z)\| \\ &\leq 16KC_{22}d_\Omega(f(z)) \quad (\text{by Lemma 6}), \end{aligned}$$

which implies that

$$(2.26) \quad \text{diam} f(B(z)) \leq 16KC_{22}d_\Omega(f(z)).$$

By (2.26) and [6, Theorem 2], we conclude that Ω is a radial John disk. The proof of this theorem is complete. \square

Proof of Theorem 3. Let $f = h + \bar{g} \in \mathcal{S}_H^0(K, \Omega)$, where Ω is a radial John disk. Assume that $a_1 = re^{i\theta}$ and $r_1e^{i\theta_1}, r_2e^{i\theta_2} \in B(a_1)$ with $r_1 \leq r_2$, where $r = |a_1|$. Since Ω is a radial John disk Ω , by [6, Theorem 1], we see that there are constants $C_{23} > 0$ and $\delta_9 \in (0, 1)$ such that for each $\zeta \in \partial\mathbb{D}$ and for $0 \leq r \leq \rho < 1$,

$$(2.27) \quad \|D_f(\rho\zeta)\| \leq C_{23}\|D_f(r\zeta)\| \left(\frac{1-\rho}{1-r} \right)^{\delta_9-1}.$$

Then, by (2.27) and Lemma B, there is a positive constant C_{24} such that

$$\begin{aligned} |f(r_2e^{i\theta_2}) - f(r_1e^{i\theta_1})| &\leq |f(r_2e^{i\theta_2}) - f(re^{i\theta_2})| + |f(r_1e^{i\theta_1}) - f(re^{i\theta_1})| \\ &\quad + |f(re^{i\theta_2}) - f(re^{i\theta_1})| \\ &\leq \int_r^{r_2} \|D_f(\rho e^{i\theta_2})\| d\rho + \int_r^{r_1} \|D_f(\rho e^{i\theta_1})\| d\rho + J \\ &\leq C_{23} \left[\int_r^{r_2} \|D_f(re^{i\theta})\| \left(\frac{1-\rho}{1-r} \right)^{\delta_9-1} d\rho \right. \\ &\quad \left. + \int_r^{r_1} \|D_f(re^{i\theta})\| \left(\frac{1-\rho}{1-r} \right)^{\delta_9-1} d\rho \right] + J \\ &\leq \frac{2C_{23}}{\delta_9} \|D_f(re^{i\theta})\| (1-r) + J, \end{aligned}$$

where

$$\begin{aligned} J &= r \int_{\gamma_0} \|D_f(re^{it})\| dt \leq C_{24}r \int_{\gamma_0} \|D_f(re^{i\theta})\| dt \\ &\leq C_{24}|\theta_2 - \theta_1| \|D_f(re^{i\theta})\| \\ &\leq 2\pi C_{24} \|D_f(re^{i\theta})\| (1-r), \end{aligned}$$

where γ_0 is the smaller subarc of $\partial\mathbb{D}_r$ between $re^{i\theta_1}$ and $re^{i\theta_2}$. Combining the last two inequalities shows that

$$|f(r_2e^{i\theta_2}) - f(r_1e^{i\theta_1})| \leq \left(\frac{2C_{23}}{\delta_9} + 2\pi C_{24} \right) \|D_f(re^{i\theta})\| (1-r)$$

Hence there is a constant $C_{25} > 0$ such that

$$(2.28) \quad \text{diam} B(a_1) \leq C_{25}(1 - |a_1|) \|D_f(a_1)\|.$$

Moreover, by Lemmas 6 and B, we see that there is a constant $C_{26} > 0$ such that

$$\begin{aligned} \text{diam} f(B(a_2)) &\geq d_\Omega(f(a_2)) \\ &\geq \frac{1}{16K} (1 - |a_2|^2) \|D_f(a_2)\| \\ &\geq \frac{1}{16K} (1 - |a_2|) \|D_f(a_2)\| \\ (2.29) \quad &\geq \frac{C_{26}}{16K} (1 - |a_2|) \|D_f(|a_2|e^{i\theta})\|. \end{aligned}$$

By (2.28), (2.29) and Lemma 5, we conclude that

$$\begin{aligned} \frac{\text{diam} f(B(a_1))}{\text{diam} f(B(a_2))} &\leq \frac{16KC_{25}}{C_{26}} \frac{(1 - |a_1|) \|D_f(a_1)\|}{(1 - |a_2|) \|D_f(|a_2|e^{i\theta})\|} \\ &\leq \frac{2^{5+\alpha}KC_{25}}{C_{26}} \frac{(1 - |a_1|)}{(1 - |a_2|)} \left(\frac{1 - |a_1|}{1 - |a_2|} \right)^{\alpha-1} \\ &= \frac{2^{5+\alpha}KC_{25}}{C_{26}} \left(\frac{1 - |a_1|}{1 - |a_2|} \right)^{\alpha} \end{aligned}$$

and the proof of the theorem is complete. \square

Lemma 7. *For $K \geq 1$, suppose that $f \in \mathcal{S}_H(K)$. Let a_1, a_2 and a_3 be positive constants and let $0 < |z_0| = 1 - \delta$, where $\delta \in (0, 1)$. Suppose further that $0 \leq 1 - a_2\delta \leq |z| \leq 1 - a_1\delta$ and $|\arg z - \arg z_0| \leq a_3\delta$. Then*

$$|f(z) - f(z_0)| \leq \frac{K}{\alpha(1+K)} \left[\left(\frac{M(a_1, a_2, a_3)}{2} \right)^{\frac{2\alpha}{1+\alpha}} - 1 \right] (1 - |z_0|^2) |f_z(z_0)|,$$

where $M(a_1, a_2, a_3)$ is defined in Lemma B.

Proof. Follows from [6, Lemma 2], but for the sake of completeness, we include certain details.

Let $\angle AOB = 2a_3\delta$ and z_1, z_2, z_3 line in the line OB with $|z_1| \leq |z_2| = |z_0| \leq |z_3|$ (see Figure 2). Clearly the distance from O to B is less than 1. Then the length of

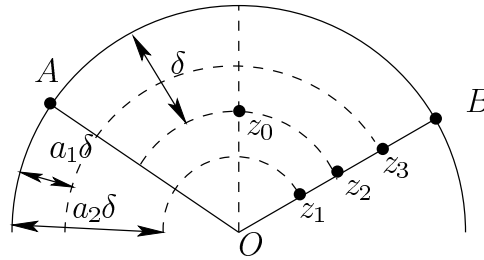


FIGURE 2

the circular arc from z_0 to z_2 is less than $a_3\delta$. As in [6, Lemma 2], it is easy to see that

$$\lambda_{\mathbb{D}}(z_0, z_2) < a_3, \quad \left| \frac{z_3 - z_1}{1 - \bar{z}_1 z_3} \right| \leq \frac{a_2 - a_1}{a_2} \quad \text{and} \quad \lambda_{\mathbb{D}}(z_0, z) \leq a_3 + \frac{1}{2} \log \frac{2a_2 - a_1}{a_1}.$$

The desired conclusion follows if we use Lemma 1. \square

The following result is an easy consequence of Lemmas 6 and 7.

Corollary 2. *Under the hypotheses of Lemma 7, we also have*

$$|f(z) - f(z_0)| \leq \frac{16K^2}{\alpha(1+K)} \left[\left(\frac{M(a_1, a_2, a_3)}{2} \right)^{\frac{2\alpha}{1+\alpha}} - 1 \right] d_{f(\mathbb{D})}(f(z_0)),$$

where $M(a_1, a_2, a_3)$ is defined in Lemma B.

Proof of Theorem 4. Let $z = re^{i\theta}$, $\sigma = |\zeta_1 - \zeta_2|$ and $\zeta_j = r_j e^{i\theta_j}$ ($j = 1, 2$) with $r_1 \leq r_2$.

Step 1. If $r \leq \rho = 1 - 2\sigma \leq r_1 \leq r_2$, then

$$\begin{aligned}
 |\zeta_1 - \zeta_2| &= |r_1 e^{i\theta_1} - r_2 e^{i\theta_2}| \\
 &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)} \\
 &= \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)} \\
 &\geq 2\sqrt{r_1 r_2} \left| \sin \frac{\theta_1 - \theta_2}{2} \right| \\
 &\geq \frac{2\rho|\theta_1 - \theta_2|}{\pi},
 \end{aligned}$$

which, together with [6, Theorem 2], Lemmas 6 and 7, imply that there are positive constants C_{27} , C_{28} , C_{29} , C_{30} , and $\delta_{10} \in (0, 1)$ such that

$$\begin{aligned}
 |f(\zeta_1) - f(\zeta_2)| &\leq |f(\zeta_1) - f(\rho e^{i\theta_1})| + |f(\zeta_2) - f(\rho e^{i\theta_2})| + |f(\rho e^{i\theta_1}) - f(\rho e^{i\theta_2})| \\
 &\leq C_{27}[(1 - \rho)\|D_f(\rho e^{i\theta_1})\| + (1 - \rho)\|D_f(\rho e^{i\theta_2})\|] \\
 &\quad + \rho \int_{\gamma_1} \|D_f(\rho e^{it})\| dt \quad (\text{by Lemma 7}) \\
 &\leq C_{27}[(1 - \rho)\|D_f(\rho e^{i\theta_1})\| + (1 - \rho)\|D_f(\rho e^{i\theta_2})\|] \\
 &\quad + \rho \int_{\gamma_1} \|D_f(z)\| \left(\frac{1 - \rho}{1 - |z|} \right)^{\delta_{10}-1} dt \quad (\text{by [6, Theorem 2]}) \\
 &\leq \|D_f(z)\| \left(\frac{1 - \rho}{1 - |z|} \right)^{\delta_{10}-1} [C_{28}(1 - \rho) + C_{29}\ell(\gamma_1)] \\
 &\leq \|D_f(z)\| \left(\frac{1 - \rho}{1 - |z|} \right)^{\delta_{10}-1} [C_{28}(1 - \rho) + \frac{C_{29}\pi\sigma}{2}] \\
 &\leq C_{30}d_f(z) \left(\frac{|\zeta_1 - \zeta_2|}{1 - |z|} \right)^{\delta_{10}} \quad (\text{by Lemma 6})
 \end{aligned}$$

where γ_1 is the smaller subarc of $\partial\mathbb{D}_\rho$ between $\rho e^{i\theta_1}$ and $\rho e^{i\theta_2}$, and

$$\ell(\gamma_1) = \rho|\theta_1 - \theta_2| \leq \frac{\pi\sigma}{2}.$$

Step 2. If $r_1 < \rho = 1 - 2\sigma$, then, by Lemma 7, there are positive constants C_{31} and C_{32} such that

$$(2.30) \quad \|D_f(\zeta)\| \leq C_{31}\|D_f(\zeta_1)\| \leq C_{32}\|D_f(\rho e^{i\theta_1})\|,$$

where $|\zeta - \zeta_1| \leq \sigma$. We see that there are positive constants C_{33} and $\delta_{11} \in (0, 1)$ such that

$$\begin{aligned}
|f(\zeta_1) - f(\zeta_2)| &\leq \int_{[\zeta_1, \zeta_2]} \|D_f(\zeta)\| |d\zeta| \\
&\leq C_{32} \|D_f(\rho e^{i\theta_1})\| |\zeta_1 - \zeta_2| \quad (\text{by (2.30)}) \\
&\leq C_{33} \|D_f(z)\| |\zeta_1 - \zeta_2| \left(\frac{1-\rho}{1-r} \right)^{\delta_{11}-1} \quad ([6, \text{Theorem 2}]) \\
&\leq 2^{3+\delta_{11}} K C_{33} d_\Omega(f(z)) \left(\frac{|\zeta_1 - \zeta_2|}{1-|z|} \right)^{\delta_{11}} \quad (\text{by Lemma 6}).
\end{aligned}$$

Step 3. If $1 - 2\sigma < r$, then, by [6, Theorem 2], we conclude that there are constants $C_{34} > 0$ and $\delta_{12} \in (0, 1)$ such that

$$|f(\zeta_1) - f(\zeta_2)| \leq 2^{\delta_{12}} C_{34} d_\Omega(f(z)) \left(\frac{|\zeta_1 - \zeta_2|}{1-|z|} \right)^{\delta_{12}}.$$

The proof of this theorem is complete. \square

The following result is an improvement of [6, Lemma 3].

Lemma 8. *Let $f \in \mathcal{S}_H(K, \Omega)$, where $G = f(\mathbb{D})$ is a bounded domain. If there are constants $C_{35} > 0$ and $\delta_{13} \in (0, 1)$ such that for each $\zeta \in \partial\mathbb{D}$ and for $0 \leq r \leq \rho < 1$,*

$$(2.31) \quad \|D_f(\rho\zeta)\| \leq C_{35} \|D_f(r\zeta)\| \left(\frac{1-\rho}{1-r} \right)^{\delta_{13}-1},$$

then, for $a \in \mathbb{D}$, we have

$$\text{diam} f(I(a)) \leq 32 K C_{36} d_G(a), \quad C_{36} = 2\pi e^{(1+\alpha)\pi} + \frac{2C_{35} e^{(1+\alpha)\pi} + C_{35}}{\delta_{13}},$$

where $I(a) = \{z \in \partial\mathbb{D} : |\arg z - \arg a| \leq \pi(1 - |a|)\}$.

Proof. For $a \in \mathbb{D}$, let $a = \rho\zeta$ with $\rho = |a|$. For $z \in I(a)$, by Lemma B, we have

$$|f(z\rho) - f(\rho\zeta)| \leq \int_{\gamma'} \rho \|D_f(\rho\xi)\| |d\xi| \leq 2e^{(1+\alpha)\pi} \rho \|D_f(\rho\zeta)\| \ell(\gamma'),$$

where γ' is the smaller subarc of $\partial\mathbb{D}_\rho$ between ρz and $\rho\zeta$, so that

$$\ell(\gamma') = \int_{\gamma'} |d\xi| = \rho |\arg(\rho\zeta) - \arg z| \leq \pi\rho(1 - \rho) \leq \pi(1 - \rho).$$

Therefore,

$$(2.32) \quad |f(z\rho) - f(\rho\zeta)| \leq 2\pi e^{(1+\alpha)\pi} (1 - \rho) \|D_f(\rho\zeta)\|.$$

Next, we have

$$\begin{aligned}
|f(z\rho) - f(z)| &\leq \int_{\rho}^1 \|D_f(tz)\| dt \\
&\leq C_{35} \int_{\rho}^1 \|D_f(\rho z)\| \left(\frac{1-t}{1-\rho}\right)^{\delta_{13}-1} dt \quad (\text{by (2.31)}) \\
&= \frac{C_{35}}{\delta_{13}} (1-\rho) \|D_f(\rho z)\| \\
(2.33) \quad &\leq \frac{2C_{35}e^{(1+\alpha)\pi}}{\delta_{13}} (1-\rho) \|D_f(\rho\zeta)\| \quad (\text{by Lemma B})
\end{aligned}$$

and, finally,

$$\begin{aligned}
|f(\zeta\rho) - f(\zeta)| &\leq \int_{\rho}^1 \|D_f(t\zeta)\| dt \\
&\leq C_{35} \int_{\rho}^1 \|D_f(\rho\zeta)\| \left(\frac{1-t}{1-\rho}\right)^{\delta_{13}-1} dt \quad (\text{by (2.31)}) \\
(2.34) \quad &= \frac{C_{35}}{\delta_{13}} (1-\rho) \|D_f(\rho\zeta)\|.
\end{aligned}$$

Again, for $z \in I(a)$, by (2.32), (2.33), (2.34) and the triangle inequality, we obtain

$$\begin{aligned}
|f(\zeta) - f(z)| &\leq |f(\rho\zeta) - f(\rho z)| + |f(z) - f(\rho z)| + |f(\rho\zeta) - f(\zeta)| \\
&\leq C_{36}(1-\rho) \|D_f(\rho\zeta)\| \\
&\leq 16KC_{36}d_G(a) \quad (\text{by Lemma 6}),
\end{aligned}$$

which in turn implies that $\text{diam} f(I(a)) \leq 32KC_{36}d_G(a)$ and the proof of the lemma is complete. \square

For $p \in (0, \infty]$, the *generalized Hardy space* $H_g^p(\mathbb{D})$ consists of all those functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that f is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup_{0 < r < 1} M_p(r, f) & \text{if } p \in (0, \infty) \\ \sup_{z \in \mathbb{D}} |f(z)| & \text{if } p = \infty \end{cases},$$

and

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

We refer to [7] for more details on $H_g^p(\mathbb{D})$.

Proof of Theorem 5. Case 1. Let $f = h + \bar{g} \in \mathcal{S}_{H_2}(K, G)$, where G is a bounded Pommerenke interior domain. Then, by definition, (1.7) holds and thus (see for example, [29, Proof of Theorem 3]), there are constants $\rho_0 \in (0, 1)$ and $\beta_1 > 0$ such that, for $\rho_0 \leq \rho < 1$ and $\theta \in [0, 2\pi]$,

$$(2.35) \quad \text{Re} \left[e^{i\theta} \frac{h''(\rho e^{i\theta})}{h'(\rho e^{i\theta})} \right] \geq -\frac{1-\beta_1}{1-\rho}.$$

For $\rho_0 \leq r \leq \rho < 1$, by integrating both sides of (2.35), we have

$$(1-r)^{\beta_1-1}|h'(re^{i\theta})| \leq (1-\rho)^{\beta_1-1}|h'(\rho e^{i\theta})|,$$

which, by (2.3), deduces that

$$(2.36) \quad (1-r)^{\beta_1-1}\|D_f(re^{i\theta})\| \leq \frac{2K}{1+K}(1-\rho)^{\beta_1-1}\|D_f(\rho e^{i\theta})\|.$$

For $\rho \in (\rho_0, 1)$, we choose a positive integer N and r_0, \dots, r_N with $r_N = \rho_0 < r_{N-1} < \dots < r_1 < r_0 = \rho$ such that, for $n \in \{0, 1, \dots, N-1\}$,

$$2^n(1-\rho) \leq 1-r_n < 2^{n+1}(1-\rho).$$

For $\theta \in [0, 2\pi)$, let

$$I(r_n e^{i\theta}) = \{\zeta \in \partial\mathbb{D} : |\arg \zeta - \theta| \leq \pi(1-r_n)\}.$$

For $2 \leq n \leq N$ and $e^{it} \in I(r_n e^{i\theta}) \setminus I(r_{n-1} e^{i\theta})$, let $\varphi = t - \theta$. Then, for $2 \leq n \leq N$,

$$(2.37) \quad \pi(1-r_{n-1}) \leq |\varphi| \leq \pi(1-r_n).$$

By the assumption, we let

$$(2.38) \quad c_p = \sup_{0 < r < 1} \left\{ \sup_{w_1, w_2 \in \gamma_r} \frac{\ell(\gamma_r[w_1, w_2])}{d_{G_r}(w_1, w_2)} \right\} < \infty,$$

where γ_r is given by (1.5). Then, by (1.8), (2.38) and [6, Theorem 4], $\|D_f\| \in H_g^1(\mathbb{D})$. Hence, for $n \in \{0, 1, \dots, N-1\}$, by (1.8), (2.38), Lemma 8 and [6, Inequality (2.3)], there is a positive constant C_{37} such that

$$\begin{aligned} \frac{1}{K} \int_{I(r_n e^{i\theta})} \|D_f(e^{it})\| dt &\leq \int_{I(r_n e^{i\theta})} l(D_f(e^{it})) dt \\ &\leq \int_{I(r_n e^{i\theta})} |df(e^{it})| = \ell(I(r_n e^{i\theta})) \\ &\leq c_p \text{diam}(f(I(r_n e^{i\theta}))) \quad (\text{by (2.38)}) \\ &\leq C_{37} c_p d_G(r_n e^{i\theta}) \quad (\text{by (1.8) and Lemma 8}) \\ (2.39) \quad &\leq \frac{2K C_{37} c_p}{1+K} (1-r_n) \|D_f(r_n e^{i\theta})\| \quad (\text{by [6, Inequality (2.3)]}). \end{aligned}$$

Let $I_n(\theta) = I(r_n e^{i\theta})$. Since $\partial\mathbb{D} = I_0(\theta) \cup (I_1(\theta) \setminus I_0(\theta)) \cap \dots \cap (I_N(\theta) \setminus I_{N-1}(\theta))$, by (2.36) and (2.37), we see that

$$(2.40) \quad \Lambda_f = \int_0^{2\pi} \|D_f(e^{it})\| \frac{1-\rho^2}{|e^{it} - \rho e^{i\theta}|^2} dt \leq J_0 + \sum_{n=1}^N J_n$$

where, by (2.39),

$$(2.41) \quad J_0 = \frac{2}{1-\rho} \int_{I_0(\theta)} \|D_f(e^{it})\| dt \leq \frac{4K^2 C_{37} c_p}{1+K} \|D_f(\rho e^{i\theta})\|$$

and

$$\begin{aligned}
J_n &= \int_{I_n(\theta) \setminus I_{n-1}(\theta)} \|D_f(e^{it})\| \frac{1 - \rho^2}{|1 - \rho e^{i(\theta-t)}|^2} dt \\
&= \int_{I_n(\theta) \setminus I_{n-1}(\theta)} \frac{\|D_f(e^{it})\| (1 - \rho^2)}{(1 - \rho)^2 + 4\rho \sin^2 \frac{\varphi}{2}} dt \\
&\leq \int_{I_n(\theta) \setminus I_{n-1}(\theta)} \|D_f(e^{it})\| \frac{\pi^2 (1 - \rho^2)}{4\rho \varphi^2} dt \\
&\leq \int_{I_n(\theta) \setminus I_{n-1}(\theta)} \frac{\|D_f(e^{it})\| (1 - \rho^2)}{4\rho (1 - r_{n-1})^2} dt \quad (\text{by (2.37)}) \\
&\leq \int_{I_n(\theta)} \frac{\|D_f(e^{it})\| (1 - \rho^2)}{4\rho (1 - r_{n-1})^2} dt \\
&\leq \frac{K^2 C_{37} c_p}{1 + K} \left(\frac{(1 - \rho)(1 - r_n) \|D_f(r_n e^{it})\|}{\rho (1 - r_{n-1})^2} \right) \quad (\text{by (2.39)}) \\
&\leq \frac{16K^2 C_{37} c_p}{(1 + K)\rho} \left(\frac{(1 - \rho) \|D_f(r_n e^{it})\|}{1 - r_n} \right).
\end{aligned}$$

By (2.36), (2.40), (2.41) and the last inequality, we conclude that

$$\begin{aligned}
\Lambda_f &\leq \frac{4K^2 C_{37} c_p}{1 + K} \|D_f(\rho e^{it})\| \left(1 + \frac{4}{\rho} \sum_{n=1}^N \frac{(1 - \rho)}{(1 - r_n)} \frac{\|D_f(r_n e^{it})\|}{\|D_f(\rho e^{it})\|} \right) \\
&\leq \frac{4K^2 C_{37} c_p}{1 + K} \|D_f(\rho e^{it})\| \left(1 + \frac{8K}{1 + K} \frac{1}{\rho} \sum_{n=1}^N \frac{1}{2^{n\beta_1}} \right) \quad (\text{by (2.36)}).
\end{aligned}$$

Thus,

$$\sup_{\zeta \in \mathbb{D} \setminus \overline{\mathbb{D}_{\rho_0}}} \frac{1}{2\pi} \int_{\partial \mathbb{D}} \frac{\|D_f(\xi)\|}{\|D_f(\zeta)\|} \frac{1 - |\zeta|^2}{|\xi - \zeta|^2} |d\xi| < \infty,$$

Case 2. For $\rho \in [0, \rho_0]$ and $\theta \in [0, 2\pi]$, by [6, Theorem 4], we have

$$\int_{\mathbb{D}} \|D_f(\xi)\| \frac{(1 - \rho^2)}{|\xi - \rho e^{i\theta}|^2} |d\xi| \leq \frac{1 + \rho_0}{1 - \rho_0} \int_{\mathbb{D}} \|D_f(\xi)\| |d\xi| < \infty.$$

On the other hand, for $\theta \in [0, 2\pi]$,

$$\min_{\rho \in [0, \rho_0]} \|D_f(\rho e^{i\theta})\| > 0.$$

In this case, Theorem 5 follows from the last two inequalities. \square

REFERENCES

1. R. BALASUBRAMANIAN, V. KARUNAKARAN AND S. PONNUSAMY, A proof of Hall's conjecture on starlike mappings, *J. London Math. Soc.* **48**(2) (1993), 278–288.
2. R. BALASUBRAMANIAN, AND S. PONNUSAMY, An alternate proof of Hall's theorem on a conformal mapping inequality, *Bull. Belg. Math. Soc.* **3**(1996), 211–215
3. A. F. BEARDON AND T. K. CARNE, Euclidean and hyperbolic lengths of images of arcs, *Proc. London Math. Soc.*, **97** (2008), 183–208.

4. T. CARROLL AND J. B. TWOMEY, Conformal mappings of close-to-convex domains, *J. London Math. Soc.*, **55** (1997), 489–498.
5. S. L. CHEN, G. LIU AND S. PONNUSAMY, Linear measure and K -quasiconformal harmonic mappings (in Chinese), *Sci. Sin. Math.*, to appear.
6. S. L. CHEN AND S. PONNUSAMY, John disks and K -quasiconformal harmonic mappings, *J. Geom. Anal.*, DOI: 10.1007/s12220-016-9727-6, Published online, 2016.
7. S. L. CHEN, S. PONNUSAMY AND A. RASILA, On characterizations of Bloch-type, Hardy-type and Lipschitz-type spaces, *Math. Z.*, **279** (2015), 163–183.
8. S. L. CHEN, S. PONNUSAMY, A. RASILA AND X. WANG, Linear connectivity, Schwarz-Pick lemma and univalence criteria for planar harmonic mappings, *Acta Math. Sinica (English Series)*, **32** (2016), 297–308.
9. M. CHUAQUI, R. HERNÁNDEZ AND M. J. MARTÍN, Affine and linear invariant families of harmonic mappings, *Math. Ann.*, DOI: 10.1007/s00208-016-1418-x, Published online, 2016.
10. M. CHUAQUI, B. OSGOOD AND CH. POMMERENKE, John domains, quasidisks, and the Nehari class, *J. Reine Angew. Math.*, **471** (1996), 77–114.
11. J. G. CLUNIE AND T. SHEIL-SMALL, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **9** (1984), 3–25.
12. P. DUREN, *Harmonic mappings in the plane*, Cambridge Univ. Press, 2004.
13. K. HAG AND P. HAG, John disks and the pre-Schwarzian derivative, *Ann. Acad. Sci. Fenn. Math.*, **26** (2001), 205–224.
14. R. R. HALL, The length of ray-images under starlike mappings, *Mathematika* **23**(1976) 147–150.
15. R. R. HALL, A conformal mapping inequality for starlike functions of order $1/2$, *Bull. London Math. Soc.* **12** (1980). 119–126.
16. J. A. JENKINS, On a result of Keogh, *J. London Math. Soc.*, **31** (1956), 391–399.
17. F. JOHN, Rotation and strain, *Comm. Pure Appl. Math.*, **14** (1961), 391–413.
18. D. KALAJ, Muckenhoupt weights and Lindelöf theorem for harmonic mappings, *Adv. Math.*, **280** (2015), 301–321.
19. V. KARUNAKARAN, Length of ray-images under conformal maps, *Proc. Amer. Math. Soc.* **87** (1983). 289–294.
20. P. B. KENNEDY, Conformal mapping of bounded domains, *J. London Math. Soc.*, **31** (1956), 332–336.
21. F. R. KEOGH, A property of bounded schlicht functions, *J. London Math. Soc.*, **29** (1954), 379–382.
22. O. LEHTO AND K. I. VIRTANEN, *Quasiconformal mappings in the plane*, Springer Verlag, 1973.
23. H. LEWY, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.*, **42** (1936), 689–692.
24. J. MARCINKIEWICZ AND A. ZYGMUND, A theorem of Lusin, *Duke Math. J.*, **4** (1938), 473–485.
25. M. MATELJEVIĆ, Distortion of harmonic functions and harmonic quasiconformal quasi-isometry, *Rev. Roumaine math. pures Appl.* **51** (2006), 711–722.
26. M. MATELJEVIĆ, Quasiconformal and quasiregular harmonic analogues of Koebe’s theorem and applications, *Ann. Acad. Sci. Fenn. Math.*, **32** (2007), 301–315.
27. M. MATELJEVIĆ, Distortion of quasiregular mappings and equivalent norms on Lipschitz-type spaces, *Abstr. Appl. Anal.*, Volume 2014, Article ID 895074, 20 pages.
28. R. NÄKKI AND J. VÄISÄLÄ, John disks, *Exposition Math.*, **9** (1991), 3–43.
29. CH. POMMERENKE, One-sided smoothness conditions and conformal mapping, *J. London Math. Soc.*, **26** (1982), 77–88.
30. CH. POMMERENKE, *Boundary behaviour of conformal maps*, Springer-Verlag, 1992.
31. S. PONNUSAMY AND A. RASILA, Planar harmonic and quasiregular mappings, Topics in Modern Function Theory (Editors. St. Ruscheweyh and S. Ponnusamy): Chapter in CMFT, RMS-Lecture Notes Series No. 19, 2013, pp. 267333.

- 32. W. RUDIN, Real and Complex Analysis. McGraw-Hill. ISBN 0-07-054234-1, 1987.
- 33. T. SHEIL-SMALL, Some conformal mapping inequalities for starlike and convex functions, *J. London Math. Soc.* **1**(2) (1969), 577–587.
- 34. T. SHEIL-SMALL, Constants for planar harmonic mappings, *J. London Math. Soc.*, **42** (1990), 237–248.
- 35. J. VÄISÄLÄ, *Lectures on n -dimensional quasiconformal mappings*, Springer-Verlag, Berlin, Heidelberg, New York, xiv, 144pp, 1971.
- 36. M. VUORINEN, *Conformal geometry and quasiregular mappings, Lecture Notes in Math.* Vol. 1319, Springer-Verlag, 1988.

S. L. CHEN, COLLEGE OF MATHEMATICS AND STATISTICS, HENGYANG NORMAL UNIVERSITY, HENGYANG, HUNAN 421008, PEOPLE'S REPUBLIC OF CHINA.

E-mail address: mathechen@126.com

S. PONNUSAMY, INDIAN STATISTICAL INSTITUTE (ISI), CHENNAI CENTRE, SETS (SOCIETY FOR ELECTRONIC TRANSACTIONS AND SECURITY), MGR KNOWLEDGE CITY, CIT CAMPUS, TARAMANI, CHENNAI 600 113, INDIA.

E-mail address: samy@isichennai.res.in, samy@iitm.ac.in